arXiv:1112.6038v1 [math.NT] 28 Dec 2011

ON LARGE GAPS BETWEEN ZEROS OF THE RIEMANN ZETA-FUNCTION

SHAOJI FENG AND XIAOSHENG WU

Abstract. Assuming the Generalized Riemann Hypothesis(GRH), we show that infinitely often consecutive non-trivial zeros of the Riemann zeta-function differ by at least 3.072 times the average spacing.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta-function. We denote the non-trivial zeros of $\zeta(s)$ as $\rho = \beta + i\gamma$. Let $\gamma \leq \gamma'$ denote consecutive ordinates of the zeros of $\zeta(s)$. The von Mangoldt formulate (see [15]) gives

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T),$$

where $N(T)$ is the number of zeros of $\zeta(s)$, $s = \sigma + it$ in the rectangle $0 \leq \sigma \leq 1, 0 \leq t \leq T$. Hence, the average size of $\gamma' - \gamma$ is $2\pi/\log\gamma$. In 1973, by studying the pair correlation of the zeros of the Riemann zeta-function, Montgomery [10] suggested that there exists arbitrarily large and small gaps between consecutive zeros of $\zeta(s)$. That is to say

$$\lambda = \limsup(\gamma' - \gamma)\frac{\log\gamma}{2\pi} = \infty \quad \text{and} \quad \mu = \liminf(\gamma' - \gamma)\frac{\log\gamma}{2\pi} = 0,$$

where γ runs over all the ordinates of the zeros of the $\zeta(s)$.

In this article, we focus on the large gaps and assume the Generalized Riemann Hypothesis (GRH) is true. This conjecture states that the non-trivial zeros of the Dirichlet L-functions are on the Re(s)=1/2 line. We obtain

Theorem 1.1 *If the Generalized Riemann Hypothesis is true, then* $\lambda > 3.072$.

Unconditionally, selberg remarked in [14] that he could prove $\lambda > 1$. Assuming RH, Mueller [12] showed that $\lambda > 1.9$, and later, by a different approach, Montgomery and Odlyzko [11] obtained $\lambda > 1.9799$. This result was then improved by Conrey, Ghosh, and Gonek [4] who obtained $\lambda > 2.337$ assuming RH and $\lambda > 2.68$ in [5] assuming GRH. Recently, by making use of the Wirtinger inequality, Hall [9] proved that there exist infinity many large gaps between the zeros on the critical line of the Riemann zeta-function greater than 2.63 times the average spacing of the zeros of Riemann zeta-function. This result implies $\lambda > 2.63$ on RH. Assuming Riemann Hypothesis, H.M.Bui, M.B.Milinovich and N.Ng proved $\lambda > 2.69$ in [3] and we obtained $\lambda > 2.7327$ in [8]. On GRH, N.Ng [13] proved in 2006 that $\lambda > 3$ and this result was improved to $\lambda > 3.033$ by Bui [2] in 2009.

2010 *Mathematics Subject Classification.* Primary 11M26; Secondary 11M06 .
Key words and phrases. gaps; Riemann zeta function; zeros.

The works of [2], [5], [13] are based on the following idea of J.Mueller [12]. Let $H : \mathbb{C} \to \mathbb{R}_{\geq 0}$ be continuous and consider the associated functions

$$\mathcal{M}_1(H,T) = \int_1^T H(\frac{1}{2}+it)dt, \tag{1.1}$$

$$m(H,T;\alpha) = \sum_{T<\gamma<2T} H(\frac{1}{2}+i(\gamma+\alpha))dt, \tag{1.2}$$

$$\mathcal{M}_2(H,T;c) = \int_{-c/L}^{c/L} m(H,T;\alpha)d\alpha, \tag{1.3}$$

where $L = \log \frac{T}{2\pi}$. Note that

$$\frac{\mathcal{M}_2(H,2T;c) - \mathcal{M}_2(H,T;c)}{\mathcal{M}_1(H,2T) - \mathcal{M}_1(H,T)} < 1 \tag{1.4}$$

implies $\lambda > \frac{c}{\pi}$.

Mueller applied this idea with $H(s) = |\zeta(s)|^2$. Let $A(s)$ denote a Dirichlet polynomial

$$A(s) = \sum_{n\leq y} a(n)n^{-s}. \tag{1.5}$$

On RH, Conrey et al. used $H(s) = |A(s)|^2$ with $a(n) = d_{2.2}(n), y = T^{1-\epsilon}$ and obtained $\lambda > 2.337$. Here $d_r(n)$ is the coefficient of n^{-s} in the Dirichlet series $\zeta(s)^r$. Later, assuming GRH, they applied (1.4) to $H(s) = |\zeta(s)A(s)|^2$ with $a(n) = 1$ and $y = (T/2\pi)^{1/2-\epsilon}$ and obtained $\lambda > 2.68$. By considering a more general coefficients $a(n)$, N.Ng [13] proved $\lambda > 3$. Actually, N.Ng chose $H(s) = |\zeta(s)A(s)|^2$ with $A(s)$ has coefficients $a_r(n) = d_r(n)p(\frac{\log n}{\log y})$. In [2], H. M. Bui chose $H(s) = A_1(s) + \zeta(s)A_2(s)$, where $A_1(s), A_2(s)$ are Dirichlet series defined by (1.5) with coefficients $a_{r_1} = d_{r_1}(n)p_1(\frac{\log n}{\log y})$ and $a_{r_2} = d_{r_2}(n)p_2(\frac{\log n}{\log y})$, and proved $\lambda > 3.033$.

In this article, we choose $H(s) = |\zeta(s)A(s)|^2$, where $A(s)$ is defined by (1.5) with $y = T^{\frac{1}{2}-\epsilon}$ and the coefficients

$$a(n) = d_r(n)P_0\Big(\frac{\log n}{\log y}\Big) + d_r^*(n)P_2\Big(\frac{\log n}{\log y}\Big), \tag{1.6}$$

for P_0, P_2 are polynomials and $r \in \mathbb{N}$. Here,

$$d_r^*(n) = \frac{1}{\log^2 y}\Lambda * \Lambda * d_r(n), \tag{1.7}$$

where $*$ is the convolution and Λ is the Mongoldt function.

Let

$$\chi(s) = 2^s\pi^{s-1}\sin\frac{1}{2}s\pi\Gamma(1-s)$$

and

$$Z(t) = (\chi(\frac{1}{2}+it))^{-\frac{1}{2}}\zeta(\frac{1}{2}+it).$$

It's well known that $Z'(t)$ has a zero between the consecutive zeros of Riemann zeta-function. Since

$$|\chi(s)| = 1 \quad \text{and} \quad \frac{\chi'(\frac{1}{2}+it)}{\chi(\frac{1}{2}+it)} \sim \log \frac{t}{2\pi},$$

we have

$$|Z'(t)| \sim \left|\zeta(\frac{1}{2}+it)\right|\left| -\frac{1}{2}\log\frac{t}{2\pi} + \frac{\zeta'}{\zeta}(\frac{1}{2}+it)\right|.$$

From (1.7), it's easy to see

$$\zeta^r(s)(\frac{\zeta'}{\zeta}(s))^2 = (\log y)^2 \sum_{n=1}^{\infty} \frac{d_r^*(n)}{n^s}$$

for $s = \sigma + it$ with $\sigma > 1$. Hence, our choice of $H(s)$ may be seen as a kind of approximation to $Z(t)^{2r-2}Z'(t)^4(\log y)^{-4}$.

We now come to the precise result. We define several functions that will appear in the following. Given P_0, P_2 are polynomials and $u \in \mathbb{Z}_{\geq 0}$, we define

$$Q_{i,u}(x) = \int_0^1 \theta^u P_i(x + \theta(1-x))d\theta, \tag{1.8}$$

for $i = 0, 2$. Given $\eta \in \mathbb{R}$ and $\overrightarrow{n} = (n_1, n_2, n_3, n_4) \in (\mathbb{Z}_{\geq 0})^4$, we define

$$l_{i_1,i_2}(\overrightarrow{n}) = \int_0^1 x^{r^2+n_1+n_2-1}(1-x)^{2r+n_3+n_4} Q_{i_1,r+n_3-1}(x)Q_{i_2,r+n_4-1}(x)dx \tag{1.9}$$

$$\begin{aligned} k_{i_1,i_2}(\overrightarrow{n}) = \int_0^1 \int_0^{1-x} & x^{r+n_1-1}(\eta^{-1}-x)^{n_2} y^{r^2+n_3-1}(1-y)^{r+n_4} \\ & \cdot P_{i_1}(x+y)Q_{i_2,r+n_4-1}(y)dydx. \end{aligned} \tag{1.10}$$

For $\overrightarrow{n} = (n_1, n_2, n_3, n_4, n_5) \in (\mathbb{Z}_{\geq 0})^5$, we define

$$\begin{aligned} h_{i_1,i_2}(\overrightarrow{n}) = & -\eta^{-1}\mathrm{B}(n_5+1, r+n_4-1) l_{i_1,i_2}(n_1, n_2, n_3-1, n_4+n_5) \\ & + \mathrm{B}(n_5+1, r+n_4-1) l_{i_1,i_2}(n_1, n_2, n_3, n_4+n_5) \\ & + \mathrm{B}(n_5+1, r+n_4) l_{i_1,i_2}(n_1, n_2, n_3-1, n_4+n_5+1), \end{aligned} \tag{1.11}$$

where $\mathrm{B}(m,n)$ is the Beta function. For $r \geq 1$, we define the constants

$$i_1 = i_1' + i_1'', \qquad i_2 = i_2' + i_2''$$

$$a_r = \prod_p ((1-p^{-1})^{r^2} \sum_{m=0}^{\infty} (\frac{\Gamma(r+m)}{\Gamma(r)m!})^2 p^{-m}),$$

$$b_r(i_1', i_2') = \sum_{\tau=0}^{\min(i_1',i_2')} C_{i_1'}^{\tau} C_{i_2'}^{\tau} \tau! r^{i_1'+i_2'-2\tau} \tag{1.12}$$

$$c_r(i_1',i_2',i_1'',i_2'') = \frac{C_{i_1}^{i_1'}C_{i_2}^{i_2'}b_r(i_1',i_2')}{(r^2+i_1'+i_2'-1)!(r+i_1''-1)!(r+i_2''-1)!}, \tag{1.13}$$

with C_m^n is the binomial coefficient. For $n \geq -2$, we also define

$$\Omega_r(i_2'',n) = \begin{cases} (-1)^{n+1}C_r^{n+2} & \text{for} \quad i_2''=0 \\ \displaystyle\sum_{j'=-2}^{\min(r-2,n)}(-1)^{j'+1}C_r^{j'+2}\Delta(n-j') & \text{for} \quad i_2''=1 \\ \displaystyle\sum_{j'=-2}^{\min(r-2,n)}(-1)^{j'+1}C_r^{j'+2}\sum_{j_1+j_2=n-j'}\Delta(j_1)\Delta(j_2) & \text{for} \quad i_2''=2 \end{cases} \tag{1.14}$$

with Δ given by

$$\begin{cases} \Delta(0)=1 \\ \Delta(j)=-1, & \text{for} \quad j\geq 1. \end{cases} \tag{1.15}$$

Since

$$\sum_{j=0}^{r}(-1)^{j+1}C_r^jP(j)=0$$

for any polynomial $P(j)$ on j, it's not difficult to see

$$\Omega_r(i_2'',n)=0 \qquad \text{for} \quad n>r-2.$$

From this definitions, we can present our result for $\mathcal{M}_1(H,T)$ and $m(H,T;\alpha)$.

Theorem 1.2 *Let $y=(\frac{T}{2\pi})^\eta$ with $0<\eta<1/2$, we have*

$$\mathcal{M}_1(H,T) \sim a_{r+1}T(\log y)^{(r+1)^2}\sum_{i_1'+i_1''=0,2}\ \sum_{i_2'+i_2''=0,2}c_r(i_1',i_2',i_1'',i_2'')\hat{l}(\eta,r,i_1',i_2',i_1'',i_2'') \tag{1.16}$$

as $T\to\infty$, where

$$\begin{aligned}\hat{l}(\eta,r,i_1',i_2',i_1'',i_2'') =&\eta^{-1}l_{i_1,i_2}(i_1',i_2',i_1'',i_2'') \\ &- l_{i_1,i_2}(i_1',i_2',i_1''+1,i_2'') \\ &- l_{i_1,i_2}(i_1',i_2',i_1'',i_2''+1).\end{aligned} \tag{1.17}$$

This is valid up to an error term which is $O(L^{-1})$ smaller than the main term.

Theorem 1.3 *Suppose $r\in\mathbb{N}$ and $y=(\frac{T}{2\pi})^\eta$ with $\eta<1/2$. The Generalized Riemann Hypothesis implies*

$$\begin{aligned} m(H,T;\alpha) \sim&\frac{a_{r+1}TL^{(r+1)^2+1}}{\pi}\mathrm{Re}\sum_{j=0}^{\infty}z^j\eta^{j+(r+1)^2+1}\Big(\frac{\hat{h}(r,j,\eta)}{j!}+\hat{k}(r,j,\eta)\Big) \\ &+\frac{L}{2\pi}\mathcal{M}_1(H,T), \end{aligned} \tag{1.18}$$

where $z = i\alpha L, |z| \ll 1,$

$$\hat{h}(r,j,\eta) = \sum_{i_1'+i_1''=0,2} \sum_{i_2'+i_2''=0,2} c_r(i_1',i_2',i_1'',i_2'')$$
$$\cdot \left(r h_{i_1,i_2}(i_1',i_2',i_1''+1,i_2''+1,j) + i_2''(r+i_2''-1) h_{i_1,i_2}(i_1',i_2',i_1''+1,i_2'',j+1)\right), \tag{1.19}$$

$$\hat{k}(r,j,\eta) = \sum_{i_1'+i_1''=0,2} \sum_{i_2'+i_2''=0,2} c_r(i_1',i_2',i_1'',i_2'')$$
$$\cdot \sum_{n=-2}^{\min(r-2,j)} \frac{\Omega_r(i_2'',n)(r+i_2''-1)!}{(j-n)!(r+i_2''+n+1)!} k_{i_1,i_2}(i_1'',j-n,i_1'+i_2',i_2''+n+2). \tag{1.20}$$

This result is valid up to an error term $O_{\epsilon,r}(TL^{(r+1)^2} + T^{1/2+\eta+\epsilon})$.

From Theorem 1.2 and Theorem 1.3, an argument similar to N.Ng [13] deduce that

$$\frac{\mathcal{M}_2(H,2T;c) - \mathcal{M}_2(H,T;c)}{\mathcal{M}_1(H,2T) - \mathcal{M}_1(H,T)} = f_r(c) + O(\epsilon),$$

where

$$f_r(c) = \frac{1}{D} \sum_{j=0}^{\infty} \frac{(-1)^j c^{2j+1}}{2^{2j}} \left(\frac{\hat{h}(r,2j,\frac{1}{2})}{(2j+1)!} + \frac{\hat{k}(r,2j,\frac{1}{2})}{2j+1} \right) + \frac{c}{\pi} + O(\epsilon) \tag{1.21}$$

and

$$D := \pi \sum_{i_1'+i_1''=0,2} \sum_{i_2'+i_2''=0,2} c_r(i_1',i_2',i_1'',i_2'') \hat{l}_{i_1,i_2}(\eta,r,i_1',i_2',i_1'',i_2'').$$

It's known that $f_r(c) < 1$ implies $\lambda \geq \frac{c}{\pi}$. We may compute (1.21) for various choices of r and $P_0(x), P_2(x)$. Choosing $c = 3.072\pi, r = 2$ and $P_0(x) = x^{30}, P_2(x) = -31.4x^{165}$, we compute the sum

$$D^{-1} \sum_{j=0}^{30} \frac{(-1)^j c^{2j+1}}{2^{2j}} \left(\frac{\hat{h}(r,2j,\frac{1}{2})}{(2j+1)!} + \frac{\hat{k}(r,2j,\frac{1}{2})}{2j+1} \right) + \frac{c}{\pi} = 0.999846...$$

by *Mathematic*. On the other hand, we may bound the terms $j > 30$. For $P_0(x) = x^{30}, P_2(x) = -31.4x^{165}$, it's easy to see $|Q_{i,u}(x)| \leq 32$ on [0,1] and $l_{i_1,i_2}(\overrightarrow{n}) \leq 32^2$. So, a direct calculation gives that $\widehat{h(\overrightarrow{n})} \leq 64 \times 4 \times 32^2$ and hence

$$\begin{aligned}
\left| \frac{1}{D} \sum_{j>30}^{\infty} \frac{(-1)^j c^{2j+1}}{2^{2j}} \frac{\widehat{h}(r,2j,\frac{1}{2})}{(2j+1)!} \right| &\leq \frac{262144c}{D} \sum_{j>30} \frac{(c/2)^{2j}}{(2j+1)!} \\
&\leq \frac{262144c}{D} \sum_{j>30} e^{-2j(\log(2j)-(\log(c/2)-1))} \\
&< \frac{262144c}{D} \frac{e^{-60(\log(60)-\log(c/2)-1)}}{2\log(60)-\log(c/2)-1} < 10^{-20},
\end{aligned}$$

where we have applied $n! > (n/e)^n$. A similar calculation establishes that

$$\left|\frac{1}{D}\sum_{j>J}^{\infty}\frac{(-1)^j c^{2j+1}}{2^{2j}}\frac{\widehat{k}(r,2j,\frac{1}{2})}{(2j+1)}\right| < 10^{-20}.$$

Thus, we conclude that $f_2(3.05\pi) < 1$ and establish Theorem 1.1. If we let $r = 2$ and $P_0(x) = x^{30}, P_2(x) = 0$, we get $f_2(3\pi) = 0.999481...$, which accords with the result of N.Ng [13].

We have deduced Theorem 1.1 from Theorem 1.2 and Theorem 1.3. Hence, the rest of the article will be devoted to establishing the result of Theorem 1.2 and Theorem 1.3. From a similar argument to the part 4 of N.Ng [13], we note

$$m(H,T;\alpha) \sim 2\mathrm{Re}I + \frac{L}{2\pi}\mathcal{M}_1(H,T) \tag{1.22}$$

with an error term $O(L^{-1})$ smaller. Here,

$$I = \sum_{k\le y}\frac{a(k)}{k}\sum_{j\le kT2\pi} b(j)e(-j/k) + O(yT^{\frac{1}{2}+\epsilon}), \tag{1.23}$$

$$b(j) = -\sum_{\substack{hmn=j\\ h\le y}} a(h)d(m)\Lambda(n)n^{i\alpha}. \tag{1.24}$$

Hence, to prove Theorem 1.2 and Theorem 1.3, it's sufficient to evaluate $\mathcal{M}_1$ and I. We will evaluate $\mathcal{M}_1$ in section 4 and I in section 5.

2. Some notation and definitions

Throughout this article we shall employ the notation

$$[m]_y := \frac{\log m}{\log y} \tag{2.1}$$

for $m, y > 0$, and we appoint that p, p_i, q, q_j always denote primes for $i, j \ge 1$. The sum

$$\sum_{a_1+\cdots+a_m\ge D} \quad \text{and} \quad \sum_{a_1+\cdots+a_m=D}$$

are always over all entire arrays $(a_1, a_2, \cdots, a_m)$ with $a_i \ge 0$. In addition, we define $j_\tau(n)$ and $\sigma_r(n)$ as in N.Ng [13],

$$j_\tau(n) = \prod_{p|n}(1 + O(p^{-\tau})) \tag{2.2}$$

for $\tau > 0$ and the constant in the O is fixed and independent of τ and

$$\sigma_r(n) = \prod_{p^\lambda \| n} d_r(p^\lambda)H_{\lambda,r}(p^{-1}) \tag{2.3}$$

with

$$H_{\lambda,r}(x) := \lambda x^{-\lambda}\int_0^x t^{\lambda-1}(1-t)^{r-1}dt.$$

Here, $p^\lambda \| n$ means $p^\lambda | n$ and $p^{\lambda+1} \nmid n$. A simple calculation by part integration shows that

$$H_{\lambda,1}(x) = 1, \qquad H_{\lambda,2}(x) = 1 - \frac{\lambda}{\lambda+1}x,$$

and for $r \geq 3$,

$$\begin{aligned} &H_{\lambda,r}(x) \\ =&(1-x)^{r-1} + \sum_{i=1}^{r-2} \frac{(r-1)\cdots(r-i)}{(\lambda+1)\cdots(\lambda+i)} x^i (1-x)^{r-i-1} + \frac{(r-1)!\lambda!}{(\lambda+r-1)!} x^{r-1}. \end{aligned}$$

From this, it's easy to see $H_{\lambda,r}(x)$ is a polynomial of x with $H_{\lambda,r}(0) = 1$, and all the coefficients of the polynomial are $O(1)$. Here, the constant of the O is only decided by r. So, we have

$$\sigma_r(p_1 \cdots p_i) = r^i + O(\sum_{\tau=1}^{i} \frac{1}{p_\tau}), \tag{2.4}$$

$$\sigma_r(p_1 \cdots p_i m) \ll \sigma_r(p_1 \cdots p_i)\sigma_r(m) + O(\sigma_r(m) \sum_{\tau=1}^{i} \frac{1}{p_\tau}) \tag{2.5}$$

with the constant of O is only decided by r and i, for $m, i \geq 1$ are integers. We now also invoke several properties of d_r which we apply repeatedly as follow:

$$\begin{aligned} &\sum_{m \leq x} d_r(m) m^{-1} \ll \log^r x, \\ &\sum_{m \leq x} d_r(m)^2 m^{-1} \ll \log^{r^2} x. \end{aligned} \tag{2.6}$$

3. Some lemmas

In this section, we present some lemmas that will be used in the following.

Lemma 3.1 (Mertens Theorem).

$$\sum_{p \leq y} \frac{\log p}{p} = \log y + O(1). \tag{3.1}$$

Lemma 3.2 *For positive integers m_1,m_2 and n,*

$$\begin{aligned} &\sum_{\substack{p_1 p_2 \cdots p_{m_1} | n \\ q_1 q_2 \cdots q_{m_2} | n}} \mu^2(p_1 \cdots p_{m_1}) \log p_1 \cdots \log p_{m_1} \mu^2(q_1 \cdots q_{m_2}) \log q_1 \cdots \log q_{m_2} \\ =& \sum_{k=0}^{\min(m_1,m_2)} C_{m_1}^k C_{m_2}^k k! \sum_{p_1 \cdots p_{m_1+m_2-k} | n} \mu^2(p_1 \cdots p_{m_1+m_2-k}) \\ &\cdot \log^2 p_1 \cdots \log^2 p_k \log p_{k+1} \cdots \log p_{m_1+m_2-k} \end{aligned} \tag{3.2}$$

where p and q runs over prime numbers, C_m^k is the binomial coefficient.

This Lemma is a generalization of Lemma 2.3 in Feng [7].

Lemma 3.3 *Let $a_i \geq 1$ be integers for $1 \leq i \leq m$, $F(x) \ll M$ on $[1, y]$ be continuous,*

$$\sum_{p_1 \cdots p_m \leq y} \frac{\log^{a_1} p_1 \cdots \log^{a_m} p_m}{p_1 \cdots p_m} \int_1^{\frac{y}{p_1 \cdots p_m}} \frac{F(p_1 \cdots p_m x) \log^k x}{x} dx$$
$$= \frac{k! \prod_{i=1}^m (a_i - 1)!}{(\sum_{i=1}^m a_i + k)!} \int_1^y \frac{F(x)(\log x)^{\sum_{i=1}^m a_i + k}}{x} dx + O(M(\log y)^{\sum_{i=1}^m a_i + k}) \tag{3.3}$$

Proof. By Lemma 3.1 and Abel summation, we may express the right side of (3.3) as the expression in Lemma 9 of Feng [7]. Then, an argument similar to the proof of Lemma 9 Feng [7] establishes the Lemma.

Lemma 3.4 (Conrey [6] Lemma 3). *Suppose that $A_j(s) = \sum_{n=1}^\infty \alpha_j(n) n^{-s}$ is absolutely convergent for $\sigma > 1$, for $1 \leq j \leq J$, and that*

$$A(s) = \sum_{n=1}^\infty \frac{\alpha(n)}{n^s} = \prod_{j=1}^J A_j(s). \tag{3.4}$$

Then for any positive integer d,

$$\sum_{n=1}^\infty \frac{\alpha(dn)}{n^s} = \sum_{d_1 \cdots d_J = d} \prod_{j=1}^J \Big(\sum_{(n, d_1 \cdots d_{j-1}) = 1}^\infty \frac{\alpha_j(n d_j)}{n_s} \Big). \tag{3.5}$$

Lemma 3.5 (N.Ng [13] Lemma 5.3). *Let $(h, k) = 1$ and $k = \prod p^\lambda > 0$. For $\alpha \in \mathbb{R}$ and $\sigma > 1$ define*

$$\mathcal{Q}(s, \alpha, h/k) = -\sum_{m,n=1}^\infty \frac{d(m)\Lambda(n)}{m^s n^{s-i\alpha}} e\Big(\frac{-mnh}{k}\Big). \tag{3.6}$$

Then $Q(s, \alpha, h/k)$ has a meromorphic continuation to the entire complex plane, If $\alpha \neq 0$, $Q(s, \alpha, h/k)$ has

(i) *at most a double pole at $s = 1$ with same principal part as*

$$k^{1-2s} \zeta^2(s) \Big(\frac{\zeta'}{\zeta}(s - i\alpha) - \mathcal{G}(s, \alpha, k) \Big),$$

where

$$\mathcal{G}(s, \alpha, k) = \sum_{p | k} \log p \Big(\sum_{a=1}^{\lambda - 1} p^{a(s-1+i\alpha)} + \frac{p^{\lambda(s-1+i\alpha)}}{1 - p^{-s+i\alpha}} - \frac{1}{p^{s-i\alpha} - 1} \Big); \tag{3.7}$$

(ii) *a simple pole at $s = 1 + i\alpha$ with residue*

$$-\frac{1}{k^{i\alpha} \phi(k)} \zeta^2(1 + i\alpha) \mathcal{R}_k(1 + i\alpha)$$

where

$$\mathcal{R}_k(s) = \prod_{p^\lambda || k} (1 - p^{-1} + \lambda(1 - p^{-s}(1 - p^{s-1})). \tag{3.8}$$

Moreover, on GRH, $\mathcal{Q}(s,\alpha,h/k)$ is regular in $\delta > 1/2$ except for these two poles.

Lemma 3.6 *Assume GRH. Let $y = (\frac{T}{2\pi})^{\eta}$ where $0 < \eta < 1/2$, $k \in \mathbb{N}$ with $k \le y$, and $\alpha \in \mathbb{R}$. Set*

$$\mathcal{Q}^{*}(s,\alpha,k) = \sum_{j=1}^{\infty} b(j) j^{-s} e(-j/k) \quad (\sigma > 1), \tag{3.9}$$

where

$$b(j) = -\sum_{\substack{hmn=j \\ h\le y}} (d_r(h)P_1([h]_y) + d_r^{*}(h)P_2([h]_y))d(m)\Lambda(n)n^{i\alpha}.$$

Then $\mathcal{Q}^{*}(s,\alpha,k)$ has an analytic continuation to $\sigma > 1/2$ except possible poles at $s = 1$ and $1 + i\alpha$. Furthermore,

$$\mathcal{Q}^{*}(s,\alpha,k) = O(y^{\frac{1}{2}} T^{\epsilon})$$

where $s = \sigma + it, 1/2 + L^{-1} \le \sigma \le 1 + L^{-1}, |t| \le T, |s-1| > 0.1$, and $|s - 1 - i\alpha| > 0.1$.

Proof. From the definition of $b(j)$, we may denote $\mathcal{Q}^{*}(s,\alpha,k) = \mathcal{Q}_1^{*}(s,\alpha,k) + \mathcal{Q}_2^{*}(s,\alpha,k)$ with obvious meaning and prove both parts satisfy the Lemma. The proof of $\mathcal{Q}_1^{*}(s,\alpha,k)$ is given by Lemma 5.6 of N.Ng [13]. We can prove $\mathcal{Q}_2^{*}(s,\alpha,k)$ similarly to Lemma 5.6 of N.Ng [13]. The only difference is we replace (5.9) of N.Ng [13] with

$$\begin{aligned} B(s,d,z) = \sum_{f_1 f_2 f_3 f_4 f_5 f_6 = d} & A_1(s, f_1; z) A_2(s, f_2, f_1) A_2(s, f_3, f_1 f_2) \\ & \times A_3(s, f_4, f_1 f_2 f_3) A_4(s, f_5, f_1 f_2 f_3 f_4) A_4(s, f_6, f_1 f_2 f_3 f_4 f_5) \end{aligned}$$

by Lemma 3.4, where

$$\begin{aligned} A_1(s, f; z) &= \chi(f) \sum_{h \le y/f} \frac{\chi(h) d_r(fh)(fh)^z}{h^s}, \\ A_2(s, f, r) &= \chi(f) L(s,\chi) \prod_{p \mid r} (1 - \chi(p)p^{-s}), \\ A_3(s, f, r) &= -\sum_{(n,r)=1} \chi(fn)\Lambda(fn)(fn)^{i\alpha} n^{-s}, \\ A_4(s, f, r) &= -\sum_{(n,r)=1} \chi(fn)\Lambda(fn) n^{-s}. \end{aligned}$$

It's obvious that $A_4(s, f, r) = A_3(s, f, r)$ for $\alpha = 0$, so, the other part of the proof is the same to Lemma 5.6 of N.Ng [13].

Lemma 3.7. *For $\alpha \in \mathbb{R}$ and $j \in \mathbb{Z}_{\ge 0}$, we have*

$$\mathcal{G}^{(j)}(1,\alpha,k) = \sum_{p \mid k} p^{i\alpha} (\log p)^{j+1} + O(C_j(k)) \tag{3.10}$$

where $\mathcal{G}(s, \alpha, k)$ is defined by (3.7) and

$$C_j(k) = \sum_{p|k} \frac{\log^j p}{p} + \sum_{p^i || k, i \geq 2} \alpha \log^j p.$$

Moreover, for $x \leq y$, we have

$$\sum_{h,k \leq x} \frac{a(h)a(k)(h,k)}{hk} C_j\left(\frac{k}{(h,k)}\right) \ll (\log x)^{r^2+2r}, \tag{3.11}$$

Proof. We remark that (3.10) is proven in Conrey [5]. Recalling the definition of $a(n)$, we may denote the left side of (3.11) as

$$\sum_{h,k \leq x} \frac{C_j(\frac{k}{(h,k)})(h,k)}{hk}(d_r(h)d_r(k) + d_r^*(h)d_r(k) + d_r(h)d_r^*(k) + d_r^*(h)d_r^*(k)).$$

Thus, we express the left side of (3.11) into four parts. The first part accords with (3.11) given by Lemma 5.7 N.Ng [13] and we now prove it's also available to the other three parts. We only give the proof of the fourth part, since the other parts can be proven similarly. The part we are considering is

$$\begin{aligned} &\sum_{h,k \leq x} \frac{d_r^*(h)d_r^*(k)(h,k)}{hk} C_j\left(\frac{k}{(h,k)}\right) \\ &\leq \sum_{h,k \leq x} \frac{d_r^*(h)d_r^*(k)}{hk}(C_j(k) + 1) \sum_{a|(h,k)} \phi(a) \\ &\leq \sum_{a \leq x} \frac{1}{a} \sum_{h,k \leq \frac{x}{a}} \frac{d_r^*(ah)d_r^*(ak)(C_j(ak) + 1)}{hk}, \end{aligned} \tag{3.12}$$

where $\phi(n)$ is the number of numbers less than n and prime to n. Recalling that

$$d_r^*(n) = \frac{1}{\log^2 y} \sum_{i_1,i_2=1}^{\infty} \sum_{p_1^{i_1} p_2^{i_2} | n} \log p_1 \log p_2 d_r\left(\frac{n}{p_1^{i_1} p_2^{i_2}}\right)$$

and

$$C_j(ak) = \sum_{p|ak} \frac{\log^j p}{p} + \sum_{p^i || ak, i \geq 2} \alpha \log^j p,$$

we find the sum in (3.12) is

$$\begin{aligned} &\ll \frac{1}{\log^4 x} \sum_{a \leq x} \frac{1}{a} \sum_{h \leq \frac{x}{a}} \frac{1}{h} \sum_{i_1,i_2,j_1,j_2=1}^{\infty} \sum_{p_1^{i_1} p_2^{i_2} | ah} \log p_1 \log p_2 d_r\left(\frac{ah}{p_1^{i_1} p_2^{i_2}}\right) \\ &\cdot \sum_{k \leq \frac{x}{a}} \frac{1}{k} \sum_{q_1^{j_1} q_2^{j_2} | ak} \log q_1 \log q_2 d_r\left(\frac{ak}{q_1^{j_1} q_2^{j_2}}\right)\left(\sum_{p|ak} \frac{\log^j p}{p} + \sum_{p^i || ak, i \geq 2} \alpha \log^j p\right). \end{aligned} \tag{3.13}$$

We divide the sum in (3.13) into five parts by the number of different elements in $\{p_1, p_2, q_1, q_2, p\}$. Not shortage of general nature, we only prove the part with any two elements are different

here, for the other parts can be proven similarly. we find the part consisted by the terms with any two elements in $\{p_1, p_2, q_1, q_2, p\}$ are different in the sum of (3.13) is

$$\ll \frac{1}{\log^4 x} \sum_{p \le x} \frac{\log^j p}{p^2} \sum_{i_1, i_2, j_1, j_2 = 1}^{\infty} \sum_{p_1^{i_1} \le x} \frac{\log p_1}{p_1^{i_1}} \sum_{p_2^{i_2} \le x} \frac{\log p_2}{p_2^{i_2}} \sum_{q_1^{j_1} \le x} \frac{\log q_1}{q_1^{j_1}} \sum_{q_2^{j_2} \le x} \frac{\log q_2}{q_2^{j_2}}$$
$$\cdot \sum_{a \le x} \frac{d_r^2(a)}{a} \sum_{h \le x} \frac{d_r(h)}{h} \sum_{k \le x} \frac{d_r(k)}{k} \ll ((\log x)^{r^2+2r}),$$

for familiar formula

$$\sum_{i \ge 2} \sum_{p^i \le x} \frac{\log^j p}{p^i} = O(1)$$

with $\forall j \ge 0$. Putting together the results establishes the lemma.

Lemma 3.8 *Suppose $r, n \in \mathbb{N}, 1 \le x, n \le \frac{T}{2\pi}$, and $F \in C^1([0,1])$. There exists an absolute constant $\tau_0 = \tau_0(r)$ such that*

$$\sum_{h \le x} \frac{d_r(nh)}{h} F([h]_x) = \frac{\sigma_r(n)(\log x)^r}{(r-1)!} \int_0^1 \theta^{r-1} F(\theta) d\theta + O(d_r(n) j_{\tau_0}(n) L^{r-1}), \tag{3.14}$$

with $j_{\tau_0}(n)$ defined by (2.2). Furthermore, suppose $a_i \ge 1$ are integers for $1 \le i \le m$, we have

$$\begin{aligned} &\sum_{h \le x} \frac{F([h]_x)}{h} \sum_{p_1 \cdots p_m | h} \log^{a_1} p_1 \cdots \log^{a_m} p_m d_r\Big(\frac{nh}{p_1 \cdots p_m}\Big) \\ =& \frac{\sigma_r(n) \prod_{i=1}^m (a_i - 1)! (\log x)^{r + \sum_{i=1}^m a_i}}{(r + \sum_{i=1}^m a_i - 1)!} \int_0^1 \theta^{r + \sum_{i=1}^m a_i - 1} F(\theta) d\theta \\ &+ O(d_r(n) j_{\tau_0}(n) L^{r + \sum_{i=1}^m a_i - 1}). \end{aligned} \tag{3.15}$$

Proof. The first identity (3.14) is given by lemma 5.8 N.Ng [13]. Changing summation order and making the variable change $h \to hp_1 \cdots p_m$ yields the left side of (3.15)

$$\begin{aligned} &= \sum_{p_1 \cdots p_m \le x} \frac{\log^{a_1} p_1 \cdots \log^{a_m} p_m}{p_1 \cdots p_m} \sum_{h \le x / p_1 \cdots p_m} \frac{F([p_1 \cdots p_m h]_x)}{h} d_r(nh) \\ &= \frac{\sigma_r(n)}{(r-1)!} \sum_{p_1 \cdots p_m \le x} \frac{\log^{a_1} p_1 \cdots \log^{a_m} p_m}{p_1 \cdots p_m} \int_1^x (\log t)^{r-1} F([p_1 \cdots p_m t]_x) \frac{dt}{t} \\ &\quad + O\Big(\sum_{p_1 \cdots p_m \le x} \frac{\log^{a_1} p_1 \cdots \log^{a_m} p_m}{p_1 \cdots p_m} d_r(n) j_{\tau_0}(n) L^{r-1} \Big). \end{aligned}$$

Then (3.15) follows by Lemma 3.1 and Lemma 3.3.

Lemma 3.9 *For $r, i \in \mathbb{N}$, $1 \le i_1 \le i$ and $g \in C^1([0,1])$, we have*

$$\sum_{n\le y} \frac{\phi(n)\sigma_r(p_1\cdots p_{i_1}n)\sigma_r(p_{i_1+1}\cdots p_i n)}{n^2} g([n]_y)$$
$$=\frac{r^i a_{r+1}(\log y)^{r^2}}{(r^2-1)!}\int_0^1 \theta^{r^2-1}g(\theta)d\theta + O((\log y)^{r^2}(\sum_{\tau=1}^{i} p_\tau^{-1} + (\log y)^{-1}).$$

Moreover, suppose $a_i \ge 1$ are integers for $1 \le i \le m$, let $1 \le i_1 < i_2 < \cdots < i_{m_1} \le m$ and $1 \le i'_1 < i'_2 < \cdots < i'_{m_2} \le m$ for $0 \le m_1, m_2 \le m$, then

$$\sum_{n\le y}\frac{\phi(n)}{n^2}g([n]_y)\sum_{p_1\cdots p_m | h}\log^{a_1} p_1\cdots\log^{a_m} p_m\sigma_r\Big(\frac{n}{p_{i_1}\cdots p_{i_{m_1}}}\Big)\sigma_r\Big(\frac{n}{p_{i'_1}\cdots p_{i'_{m_2}}}\Big)$$
$$\sim \frac{r^{2m-m_1-m_2}a_{r+1}\prod_{i=1}^m(a_i-1)!(\log y)^{r^2+\sum_{i=1}^m a_i}}{(r^2+\sum_{i=1}^m a_i - 1)!}\int_0^1 \theta^{r^2+\sum_{i=1}^m a_i - 1}g(\theta)d\theta$$

plus an error $O((\log y)^{-1})$ smaller.

Proof. We remark that the first identity is a generalization of Lemma 5.9 (i) in N.Ng, and it can be proven similarly equal to

$$\frac{\sigma_r(p_1\cdots p_{i_1})\sigma_r(p_{i_1+1}\cdots p_i)a_{r+1}(\log y)^{r^2}}{(r^2-1)!}\int_0^1\theta^{r^2-1}g(\theta)d\theta$$
$$+ O((\log y)^{r^2}(\sum_{\tau=1}^{i}p_\tau^{-1} + (\log y)^{-1}),$$

then the first identity follows by (2.4). The second identity can be proven by the first identity with an argument as the proof of (3.15) in Lemma 3.8.

We define $f(k) = \mathcal{R}_k(1+i\alpha)/\phi(k)$ and $\mathcal{T}_{k;N}(\alpha) = \sum_{j=0}^N \mathcal{R}_k^{(j)}(1)(i\alpha)^j/j!$ with $\mathcal{R}_k(s)$ given by (3.8).

Lemma 3.10 (N.Ng [13] Lemma 5.11). *For $l = \log x$, $|\alpha| \ll (\log x)^{-1}$, $1 \le x, m \le y$, n square free and $n \mid m$, we have*

$$\sum_{k\le x} d_r(mk)f(nk) \ll \frac{d_r(m)j_{\tau_0}(m)l^r}{n^{1-\epsilon}}, \tag{3.16}$$

where $\tau_0 = 1/3$ is valid.

Lemma 3.11 *Let $l = \log x$, $|\alpha| \ll (\log x)^{-1}$, $\tau_0 = 1/3$ and $g \in C^1([0,1])$. We have*

$$\sum_{k\le x} d_r(mk)g([k]_x)\frac{\mathcal{R}_{nk}^{(j)}(1)}{\phi(nk)}$$
$$=\frac{\sigma_r(m)(-1)^j C_r^j(\log x)^{r+j}}{n(r+j-1)!}\int_0^1\theta^{r+j-1}g(\theta)\frac{dt}{t} + O\Big(\frac{d_r(m)j_{\tau_0}(m)l^{r+j-1}}{n^{1-\epsilon}}\Big) \tag{3.18}$$

and

$$\sum_{k\le x} d_r(mk)\Big(f(nk)-\frac{\mathcal{T}_{nk;r}(\alpha)}{\phi(nk)}\Big)\ll |\alpha|^{r+1}l^{2r}\frac{d_r(m)j_{\tau_0}(v)}{n^{1-\epsilon}}. \tag{3.19}$$

Still, suppose $a_i\ge 1$ are integers for $1\le i\le\tau$, $k=p_1\cdots p_\tau k'$, then

$$\begin{aligned}&\sum_{k\le x}\frac{g([k]_x)}{k}\sum_{p_1\cdots p_\tau|k}\log^{a_1}p_1\cdots\log^{a_\tau}p_\tau d_r(mk')\frac{nk'\mathcal{R}^{(j)}_{nk'}(1)}{\phi(nk')}\\ =&\frac{\sigma_r(m)(-1)^jC_r^j\prod_{i=1}^{\tau}(a_i-1)!(\log x)^{r+\sum_{i=1}^{\tau}a_i+j}}{(r+\sum_{i=1}^{\tau}a_i+j-1)!}\int_0^1\theta^{r+\sum_{i=1}^{\tau}a_i+j-1}g(\theta)d\theta\\ &+O\Big(\frac{d_r(m)j_{\tau_0}(m)l^{r+\sum_{i=1}^{\tau}a_i+j-1}}{n^{1-\epsilon}}\Big)\end{aligned} \tag{3.20}$$

The identities (3.18) and (3.19) are given by N.Ng [13], and the identity (3.20) can be proven by (3.18) with an argument as the proof of (3.15) in Lemma 3.8.

Lemma 3.12 *Let $A(s)=\sum_{n\le y}\frac{a(n)}{n^s}$, where $y=(\frac{T}{2\pi})^\eta$ and $\eta\in(0,\frac{1}{2})$. Then for $1\le t\le T$,*

$$\int_0^t|\zeta A(\frac{1}{2}+iu)|^2du=t\sum_{h,k\le y}\frac{a(h)a(k)(h,k)}{hk}\log\frac{t(h,k)^2e^{2\gamma-1}}{2\pi hk}+O(T), \tag{3.21}$$

here γ is Euler's constant.

This lemma is a special case of a formula of Balasubramanian, Conrey and Heath-Brown [1].

4. Evaluation of $\mathcal{M}_1$

From (1.1) we recall that

$$\mathcal{M}_1(H,T)=\int_1^T|\zeta A(\frac{1}{2}+it)|^2dt.$$

Then by Lemma 3.12,

$$\mathcal{M}_1(H,T)=T\sum_{h,k\le y}\frac{a(h)a(k)(h,k)}{hk}\log\frac{T(h,k)^2e^{2\gamma-1}}{2\pi hk}+O(T).$$

To estimate the sum we apply the Möbius inversion formula

$$f((h,k))=\sum_{\substack{m|h\\ m|k}}\sum_{n|m}\mu(n)f(\frac{m}{n}),$$

and obtain

$$\mathcal{M}_1(H,T)=T\sum_{h,k\le y}\frac{a(h)a(k)}{hk}\sum_{\substack{m|h\\ m|k}}\sum_{n|m}\frac{\mu(n)m}{n}\log\frac{Te^{2\gamma-1}m^2}{2\pi n^2hk}+O(T).$$

Changing the order of summation and replacing h by hm, k by km, we find that

$$\mathcal{M}_1(H,T) = T\sum_{m\le y}\frac{1}{m}\sum_{n|m}\frac{\mu(n)}{n}\sum_{h,k\le y/m}\frac{a(mh)a(mk)}{hk}\log\frac{Te^{2\gamma-1}}{2\pi n^2 hk} + O(T).$$

We next replace the logarithm term by $\log(T/(2\pi hk))$ with an error $O(\log n)$. A calculation shows that this $O(\log n)$ term contributes $O(TL^{r^2+2r})$ in $\mathcal{M}_1(H,T)$. Since $\sum_{n|m}\mu(n)n^{-1} = \phi(m)m^{-1}$ we deduce that

$$\mathcal{M}_1(H,T) = T\sum_{m\le y}\frac{\phi(m)}{m^2}\sum_{h,k\le y/m}\frac{a(mh)a(mk)}{hk}\log\frac{T}{2\pi hk} + O(TL^{r^2+2r}).$$

Recalling the definition of $a(n)$, we denote

$$\begin{aligned}\mathcal{M}_1(H,T) =& T\sum_{m\le y}\frac{\phi(m)}{m^2}\sum_{h,k\le y/m}\frac{\log\frac{T}{2\pi hk}}{hk}(d_r(mh)d_r(mk)P_1([mh]_y)P_1([mk]_y)\\ &+d_r^*(mh)d_r(mk)P_2([mh]_y)P_1([mk]_y)\\ &+d_r(mh)d_r^*(mk)P_1([mh]_y)P_2([mk]_y)\\ &+d_r^*(mh)d_r^*(mk)P_2([mh]_y)P_2([mk]_y)) + O(TL^{r^2+2r})\\ =&\mathcal{M}_{11}+\mathcal{M}_{12}+\mathcal{M}_{13}+\mathcal{M}_{14}+O(TL^{r^2+2r})\end{aligned}$$

with obvious meaning. We now come to calculate $\mathcal{M}_{14}$. Recalling the definition of $d_r^*(n)$ by (1.7), we observe that

$$d_r^*(mh) =\frac{1}{\log^2 y}\sum_{i_1,i_2=1}^{\infty}\sum_{p_1^{i_1}p_2^{i_2}|mh}\log p_1\log p_2 d_r\Big(\frac{mh}{p_1^{i_1}p_2^{i_2}}\Big),$$

$$d_r^*(mk) =\frac{1}{\log^2 y}\sum_{j_1,j_2=1}^{\infty}\sum_{q_1^{j_1}q_2^{j_2}|mk}\log q_1\log q_2 d_r\Big(\frac{mk}{q_1^{j_1}q_2^{j_2}}\Big).$$

We may replace $d_r^*(mh)$ and $d_r^*(mk)$ in $\mathcal{M}_{14}$ by

$$\frac{1}{\log^2 y}\sum_{p_1p_2|mh}\mu^2(p_1p_2)\log p_1\log p_2 d_r\Big(\frac{mh}{p_1p_2}\Big)$$

and

$$\frac{1}{\log^2 y}\sum_{q_1q_2|mk}\mu^2(p_1p_2)\log q_1\log q_2 d_r\Big(\frac{mk}{q_1q_2}\Big)$$

respectively, the error in calculation of $\mathcal{M}_1(H,T)$ caused by this is from the terms with $\max(i_1,i_2,j_1,j_2)\ge 2$ and the terms with $p_1=p_2$ or $q_1=q_2$. Since

$$\sum_{i\ge 2}\sum_{p^i\le y}\frac{\log^a p}{p^i} = O(1),$$

for $\forall a \ge 0$, we have the sum of the terms with $\max(i_1, i_2, j_1, j_2) \ge 2$

$$\begin{aligned}
&\ll TL^{-3} \sum_{\max(i_1,i_2,j_1,j_2)\ge 2} \sum_{m\le y} \frac{1}{m} \sum_{h,k\le y/m} \frac{1}{hk} \sum_{p_1^{i_1} p_2^{i_2} | mh} \log p_1 \log p_2 d_r\Big(\frac{mh}{p_1^{i_1} p_2^{i_2}}\Big) \\
&\quad \cdot \sum_{q_1^{j_1} q_2^{j_2} | mk} \log q_1 \log q_2 d_r\Big(\frac{mk}{q_1^{j_1} q_2^{j_2}}\Big) \\
&\ll TL^{-3} \sum_{i_1\ge 2} \sum_{p_1^{i_1}\le y} \frac{\log p_1}{p_1^{i}} \sum_{i_2\ge 1} \sum_{p_2\le y} \frac{\log p_2}{p_2^{i_2}} \sum_{j_1\ge 1} \sum_{q_1\le y} \frac{\log q_1}{q_1^{j_1}} \sum_{j_2\ge 1} \sum_{q_2\le y} \frac{\log q_2}{q_2^{j_2}} \\
&\quad \cdot \sum_{m\le y} \frac{1}{m} \sum_{h,k\le y/m} \frac{d_r(mh) d_r(mk)}{hk} \\
&= O(TL^{r^2+2r})
\end{aligned}$$

and the sum of the terms with $p_1 = p_2$

$$\begin{aligned}
&\ll TL^{-3} \sum_{i_1,i_2,j_1,j_2\ge 1} \sum_{m\le y} \frac{1}{m} \sum_{h,k\le y/m} \frac{1}{hk} \sum_{p_1^{i_1+i_2} | mh} \log^2 p_1 d_r\Big(\frac{mh}{p_1^{i_1+i_2}}\Big) \\
&\quad \cdot \sum_{q_1^{j_1} q_2^{j_2} | mk} \log q_1 \log q_2 d_r\Big(\frac{mk}{q_1^{j_1} q_2^{j_2}}\Big) \\
&= O(TL^{r^2+2r}).
\end{aligned}$$

This is also valid to the terms with $q_1 = q_2$. So

$$\begin{aligned}
\mathcal{M}_{14} = {} & T(\log y)^{-4} \sum_{m\le y} \frac{\phi(m)}{m^2} \\
& \cdot \sum_{h,k\le y/m} \frac{\log \frac{T}{2\pi hk} P_2([mh]_y) P_2([mk]_y)}{hk} \sum_{p_1 p_2 | mh} \mu^2(p_1 p_2) \log p_1 \log p_2 \\
& \cdot d_r\Big(\frac{mh}{p_1 p_2}\Big) \sum_{q_1 q_2 | mk} \mu^2(q_1 q_2) \log q_1 \log q_2 d_r\Big(\frac{mk}{q_1 q_2}\Big) + O(TL^{r^2+2r}).
\end{aligned}$$

We may also replace the sums

$$\sum_{p_1 p_2 | mh} \mu^2(p_1 p_2) \log p_1 \log p_2 d_r\Big(\frac{mh}{p_1 p_2}\Big),$$

$$\sum_{q_1 q_2 | mk} \mu^2(q_1 q_2) \log q_1 \log q_2 d_r\Big(\frac{mk}{q_1 q_2}\Big)$$

by

$$\sum_{\substack{i_1'+i_1''=2\\ i_1',i_1''\geq 0}} C_2^{i_1'} \sum_{p_1\cdots p_{i_1'}|m} \mu^2(p_1\cdots p_{i_1'})\log p_1\cdots\log p_{i_1'}$$

$$\times \sum_{p_{i_1'+1}\cdots p_{i_1'+i_1''}|h} \log p_{i_1'+1}\cdots\log p_{i_1'+i_1''} d_r\Big(\frac{mh}{p_1p_2}\Big),$$

$$\sum_{\substack{i_2'+i_2''=2\\ i_2',i_2''\geq 0}} C_2^{i_2'} \sum_{q_1\cdots q_{i_2'}|m} \mu^2(q_1\cdots q_{i_2'})\log q_1\cdots\log q_{i_2'}$$

$$\times \sum_{q_{i_2'+1}\cdots q_{i_2'+i_2''}|k} \log q_{i_2'+1}\cdots\log q_{i_2'+i_2''} d_r\Big(\frac{mk}{q_1q_2}\Big)$$

respectively in $\mathcal{M}_{14}$ with an error $O(TL^{r^2+2r})$ as before. Then, we have

$$\mathcal{M}_{14} = T(\log y)^{-4} \sum_{\substack{i_1'+i_1''=2\\ i_1',i_1''\geq 0}} \sum_{\substack{i_2'+i_2''=2\\ i_2',i_2''\geq 0}} C_2^{i_1'}C_2^{i_2'} \sum_{m\leq y}\frac{\phi(m)}{m^2}$$

$$\sum_{p_1\cdots p_{i_1'}|m} \mu^2(p_1\cdots p_{i_1'})\log p_1\cdots\log p_{i_1'} \sum_{q_1\cdots q_{i_2'}|m} \mu^2(q_1\cdots q_{i_2'})\log q_1\cdots\log q_{i_2'}$$

$$\sum_{h\leq y/m} \frac{d_r(\frac{mh}{p_1p_2})}{h} P_2([mh]_y) \sum_{p_{i_1'+1}\cdots p_{i_1'+i_1''}|h} \log p_{i_1'+1}\cdots\log p_{i_1'+i_1''}$$

$$\sum_{k\leq y/m} \frac{d_r(\frac{mk}{q_1q_2})}{k} P_2([mk]_y) \sum_{q_{i_2'+1}\cdots q_{i_2'+i_2''}|k} \log q_{i_2'+1}\cdots\log q_{i_2'+i_2''} \log\frac{T}{2\pi hk}$$

plus an error $O(TL^{r^2+2r})$. We apply Lemma 3.8 to the sum over h and k to obtain

$$\mathcal{M}_{14} = \sum_{\substack{i_1'+i_1''=2\\ i_1',i_1''\geq 0}} \sum_{\substack{i_2'+i_2''=2\\ i_2',i_2''\geq 0}} \frac{T(\log y)^{2r+i_1''+i_2''-3}C_2^{i_1'}C_2^{i_2'}}{(r+i_1''-1)!(r+i_2''-1)!} \sum_{m\leq y}\frac{\phi(m)}{m^2}$$

$$\sum_{p_1\cdots p_{i_1'}|m} \mu^2(p_1\cdots p_{i_1'})\log p_1\cdots\log p_{i_1'} \sum_{q_1\cdots q_{i_2'}|m} \mu^2(q_1\cdots q_{i_2'})\log q_1\cdots\log q_{i_2'}$$

$$\sigma_r\Big(\frac{m}{p_1\cdots p_{i_1'}}\Big)\sigma_r\Big(\frac{m}{q_1\cdots q_{i_2'}}\Big)G([m]_y) + \epsilon_1+\epsilon_2+\epsilon_3 + O(TL^{r^2+2r}) \tag{4.1}$$

where

$$\begin{aligned} G(\alpha) =&\eta^{-1}(1-\alpha)^{2r+i_1''+i_2''}Q_{2,r+i_1''-1}(\alpha)Q_{2,r+i_2''-1}(\alpha)\\ &-(1-\alpha)^{2r+i_1''+i_2''+1}Q_{2,r+i_1''}(\alpha)Q_{2,r+i_2''-1}(\alpha)\\ &-(1-\alpha)^{2r+i_1''+i_2''+1}Q_{2,r+i_1''-1}(\alpha)Q_{2,r+i_2''}(\alpha)\end{aligned}$$

and

$$\epsilon_1 \ll TL^{-4}\sum_{m\le y}\frac{\sigma_r(m)L^{r+i_1'+i_2'+1}}{m}j_{\tau_0}(m)d_r(m)L^{r+i_1''+i_2''-1}$$

$$\epsilon_2 \ll TL^{-4}\sum_{m\le y}\frac{j_{\tau_0}(m)d_r(m)L^{r+i_1''+i_2''-1}}{m}\sigma_r(m)L^{r+i_1'+i_2'+1}$$

$$\epsilon_3 \ll TL^{-4}\sum_{m\le y}\frac{j_{\tau_0}(m)d_r(m)L^{r+i_1''+i_2''}}{m}j_{\tau_0}(m)d_r(m)L^{r+i_1''+i_2''-1}$$

by (2.4), (2.5), Lemma 3.1, Lemma 3.2 and an argument as before. Since

$$|\sigma_r(m)| \ll d_r(m)j_\tau(m) \quad for \quad 0<\tau\le 1$$

(see (5.13) of N.Ng [13]), it follows that

$$\epsilon_1 \ll TL^{2r}\sum_{m\le y}\frac{d_r(m)^2 j_1(m)j_{\tau_0}(m)}{m} \ll TL^{r^2+2r}.$$

A similar calculation gives $\epsilon_2, \epsilon_3 \ll TL^{r^2+2r}$. Using Lemma 3.2, we have the sum over m in (4.1) is

$$\begin{aligned}
&\sum_{m\le y}\frac{\phi(m)}{m^2}\sum_{\tau=0}^{\min(i_1',i_2')}C_{i_1'}^{\tau}C_{i_2'}^{\tau}\tau! \\
&\cdot\sum_{p_1\cdots p_{i_1'+i_2'-\tau}|m}\mu^2(p_1\cdots p_{i_1'+i_2'-\tau})\log^2 p_1\cdots\log^2 p_\tau \log p_{\tau+1}\cdots\log p_{i_1'+i_2'-\tau} \\
&\cdot\sigma_r\Big(\frac{m}{p_1\cdots p_{i_1'}}\Big)\sigma_r\Big(\frac{m}{p_1\cdots p_\tau p_{i_1'+1}\cdots p_{i_1'+i_2'-\tau}}\Big)G([m]_y) \\
\sim&(\log y)^{i_1'+i_2'}\sum_{\tau=0}^{\min(i_1',i_2')}\frac{r^{i_1'+i_2'-2\tau}a_{r+1}}{(r^2+i_1'+i_2'-1)!}\int_0^1\alpha^{r^2+i_1'+i_2'-1}G(\alpha)d\alpha
\end{aligned}$$

plus an error $O(L^{-1})$ smaller by Lemma 3.9. Employing this in (4.1), we have

$$\mathcal{M}_{14}\sim\sum_{\substack{i_1'+i_1''=2\\ i_1',i_1''\ge 0}}\sum_{\substack{i_2'+i_2''=2\\ i_2',i_2''\ge 0}}
\frac{T(\log y)^{(r+1)^2}C_2^{i_1'}C_2^{i_2'}a_{r+1}b_r(i_1',i_2')}{(r+i_1''-1)!(r+i_2''-1)!(r^2+i_1'+i_2'-1)!}\hat{I}_{2,2}(\eta,r,i_1',i_2',i_1'',i_2'')$$

with an error $O(TL^{r^2+2r})$. Here, $\hat{l}_{i_1,i_2}(\eta, r, i_1', i_1'', i_2', i_2'')$ is given by (1.17). By similar arguments, we can evaluate $\mathcal{M}_{11}$, $\mathcal{M}_{12}$, $\mathcal{M}_{13}$, and have

$$\mathcal{M}_1(H,T) \sim \sum_{i_1=0,2} \sum_{i_2=0,2} \sum_{\substack{i_1'+i_1''=i_1\\ i_1',i_1''\geq 0}} \sum_{\substack{i_2'+i_2''=i_2\\ i_2',i_2''\geq 0}}$$

$$\frac{T(\log y)^{(r+1)^2} C_2^{i_1'} C_2^{i_2'} a_{r+1} b_r(i_1', i_2')}{(r+i_1''-1)!(r+i_2''-1)!(r^2+i_1'+i_2'-1)!} \hat{l}_{i_1,i_2}(\eta, r, i_1', i_2', i_1'', i_2''). \tag{4.2}$$

This proves Theorem 1.2.

5. Evaluation of I

In this section, we will evaluate I in two steps. First, we apply the lemmas to manipulate I into a suitable form for evaluation and express $I = I_1 + I_2 + O(yT^{\frac{1}{2}+\epsilon} + TL^{(r+1)^2})$. Then, we evaluate I_1 in section 5.1 and I_2 in section 5.2 respectively. Recall that by (1.23),

$$I = \sum_{k\leq y} \frac{a(k)}{k} \sum_{j\leq \frac{kT}{2\pi}} b(j)e(-j/k) + O(yT^{\frac{1}{2}+\epsilon}). \tag{5.1}$$

Using Perron's formula with $c = 1 + L^{-1}$, the inner sum is

$$\sum_{j\leq \frac{kT}{2\pi}} b(j)e(-j/k) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} Q^*(s,\alpha,k) \left(\frac{kT}{2\pi}\right)^s \frac{ds}{s} + O(kT^{\epsilon}),$$

where $Q^*(s,\alpha,k) = \sum_{j=0}^{\infty} b(j) j^{-s} e(-j/k)$. Pulling the contour left to $c_0 = \frac{1}{2} + L^{-1}$, we have

$$\begin{aligned}\sum_{j\leq \frac{kT}{2\pi}} b(j)e(-j/k) =& \frac{1}{2\pi i}\left(\int_{c-iT}^{c_0-iT} + \int_{c_0-iT}^{c_0+iT} + \int_{c_0+iT}^{c+iT} \right) Q*(s,\alpha,k) \left(\frac{kT}{2\pi}\right)^s \frac{ds}{s} \\ &+ R_1 + R_{1+i\alpha},\end{aligned} \tag{5.2}$$

where R_u is the residue at $s = u$. By Lemma 3.6 the left and horizontal edges contribute $yT^{\frac{1}{2}+\epsilon}$. Moreover by (1.24) it follows that

$$Q^*(s,\alpha,k) = \sum_{h\leq y} \frac{a(h)}{h^s} Q(s,\alpha,h/k),$$

where $Q(s,\alpha,h/k)$ is defined by (3.6). Let $H = h/(h,k)$, $K = k/(h,k)$, then $\frac{h}{k} = \frac{H}{K}$ and $(H,K) = 1$. We deduce

$$R_1 = \sum_{h\leq y} a(h) \operatorname*{res}_{s=1} \left(Q(s,\alpha,H/K) \left(\frac{TK}{2\pi H}\right)^s s^{-1} \right).$$

By Lemma 3.5(i),

$$R_1 = K\sum_{h\le y} a(h)\operatorname*{res}_{s=1}\Big(\zeta^2(s)\Big(\frac{\zeta'}{\zeta}(s-i\alpha)-\mathcal{G}(s,\alpha,K)\Big)\Big(\frac{T}{2\pi HK}\Big)^s s^{-1}\Big)$$

$$= \frac{T}{2\pi}\sum_{h\le y}\frac{a(h)}{H}$$

$$\Big(((\zeta'/\zeta)(\overline{\tau})-\mathcal{G}(1,\alpha,K))\log\Big(\frac{Te^{2\gamma-1}}{2\pi HK}\Big)+((\zeta'/\zeta)'(\overline{\tau})-\mathcal{G}'(1,\alpha,K))\Big), \tag{5.3}$$

where $\tau = 1+i\alpha$ and $\mathcal{G}(s,\alpha,K)$ given by (3.7). Similarly, Lemma 3.5(ii) implies

$$R_{1+i\alpha} = \sum_{h\le y} a(h)\operatorname*{res}_{s=\tau}\Big(\mathcal{Q}(s,\alpha,H/K)\Big(\frac{TK}{2\pi H}\Big)^s s^{-1}\Big)$$

$$= -\frac{T}{2\pi}\frac{\zeta^2(\tau)}{\tau}\sum_{h\le y}\frac{a(h)}{H}\Big(\frac{T}{2\pi H}\Big)^{i\alpha}\frac{K\mathcal{R}_K(\tau)}{\phi(K)}. \tag{5.4}$$

Combining (5.1), (5.2), (5.3) and (5.4), we obtain

$$I = \frac{T}{2\pi}\sum_{h,k\le y}\frac{a(h)a(k)(h,k)}{hk}\Big(\log\frac{Te^{2\gamma-1}}{2\pi HK}((\zeta'/\zeta)(\overline{\tau})-\mathcal{G}(1,\alpha,K))+(\zeta'/\zeta)'(\overline{\tau})$$

$$-\mathcal{G}'(1,\alpha,K)-\frac{\zeta^2(\tau)}{\tau}\Big(\frac{T}{2\pi H}\Big)^{i\alpha}\frac{K\mathcal{R}_K(\tau)}{\phi(K)}\Big)+O(yT^{\frac12+\epsilon}).$$

From (3.10), we may write $\mathcal{G}^{(j)}(1,\alpha,K) = \sum_{p|K} p^{i\alpha}\log^{j+1} p + O(C_j(K))$, for $j = 0,1$. By Lemma 3.7, the $O(C_j(K))$ term contributes $O(TL^{(r+1)^2})$. Whence

$$I = \frac{T}{2\pi}\sum_{h,k\le y}\frac{a(h)a(k)(h,k)}{hk}\Big(\log\frac{Te^{2\gamma-1}}{2\pi HK}((\zeta'/\zeta)(\overline{\tau})-\sum_{p|K}p^{i\alpha}\log p)$$

$$+(\zeta'/\zeta)'(\overline{\tau})-\sum_{p|K}p^{i\alpha}\log^2 p-\frac{\zeta^2(\tau)}{\tau}\Big(\frac{T}{2\pi H}\Big)^{i\alpha}\frac{K\mathcal{R}_K(\tau)}{\phi(K)}\Big)$$

plus an error $O(yT^{\frac12+\epsilon}+TL^{(r+1)^2})$ with $\tau = 1+i\alpha$. It follows that

$$I = \frac{T}{2\pi}\sum_{h,k\le y}\frac{a(h)a(k)}{hk}\sum_{\substack{m|h\\ m|k}} m\sum_{n|m}\frac{\mu(n)}{n}$$

$$\cdot\Big(\log\frac{Te^{2\gamma-1}m^2}{2\pi hkn^2}((\zeta'/\zeta)(\overline{\tau})-\sum_{p|\frac{nk}{m}}p^{i\alpha}\log p)+(\zeta'/\zeta)'(\overline{\tau})-\sum_{p|\frac{nk}{m}}p^{i\alpha}\log^2 p$$

$$-\frac{\zeta^2(\tau)}{\tau}\Big(\frac{Tm}{2\pi nh}\Big)^{i\alpha}\Big(\frac{nk}{m}\Big)\frac{\mathcal{R}_{\frac{nk}{m}}(\tau)}{\phi(\frac{nk}{m})}\Big)+O(yT^{\frac12+\epsilon}+TL^{(r+1)^2}).$$

by inserting the identity

$$f((h,k)) = \sum_{\substack{m|h\\ m|k}}\sum_{n|m}\mu(n)f\Big(\frac{m}{n}\Big).$$

Interchanging summation order and making the variable changes $h \to hm$, $k \to km$ yields

$$I = \frac{T}{2\pi} \sum_{m \le y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h,k \le \frac{y}{m}} \frac{a(mh)a(mk)}{hk}$$
$$\Big(\log \frac{Te^{2\gamma-1}}{2\pi hkn^2}((\zeta'/\zeta)(\overline{\tau}) - \sum_{p|nk} p^{i\alpha} \log p) + (\zeta'/\zeta)'(\overline{\tau})$$
$$- \sum_{p|nk} p^{i\alpha} \log^2 p - \frac{\zeta^2(\tau)}{\tau} \Big(\frac{T}{2\pi nh}\Big)^{i\alpha} \frac{nk\mathcal{R}_{nk}(\tau)}{\phi(nk)}\Big) + O(yT^{\frac{1}{2}+\epsilon} + TL^{(r+1)^2}).$$

Rearrange this as $I = I_1 + I_2 + O(yT^{\frac{1}{2}+\epsilon} + TL^{(r+1)^2})$ where

$$I_1 = \frac{T}{2\pi} \sum_{m \le y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h,k \le \frac{y}{m}} \frac{a(mh)a(mk)}{hk}$$
$$\cdot \Big(- \log \frac{Te^{2\gamma-1}}{2\pi hkn^2} \sum_{p|nk} p^{i\alpha} \log p - \sum_{p|nk} p^{i\alpha} \log^2 p \Big),$$

$$I_2 = \frac{T}{2\pi} \sum_{m \le y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h,k \le \frac{y}{m}} \frac{a(mh)a(mk)}{hk}$$
$$\cdot \Big(\log \frac{Te^{2\gamma-1}}{2\pi hkn^2} \frac{\zeta'}{\zeta}(\overline{\tau}) + \Big(\frac{\zeta'}{\zeta}\Big)'(\overline{\tau}) - \frac{\zeta^2(\tau)}{\tau} \Big(\frac{T}{2\pi nh}\Big)^{i\alpha} \frac{nk\mathcal{R}_{nk}(\tau)}{\phi(nk)}\Big). \tag{5.5}$$

The first sum is

$$I_1 = \frac{T}{2\pi} \sum_{m \le y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n} \sum_{h,k \le \frac{y}{m}} \frac{a(mh)a(mk)}{hk}$$
$$\cdot \Big(- \log \frac{Te^{2\gamma-1}}{2\pi hk} \sum_{p|k} p^{i\alpha} \log p - \sum_{p|k} p^{i\alpha} \log^2 p + O(L \log n)\Big).$$

A calculation shows that the $O(L \log n)$ contributes $O(TL^{(r+1)^2})$. Then we deduce that

$$I_1 = \frac{T}{2\pi} \sum_{m \le y} \frac{\phi(m)}{m^2} \sum_{h,k \le \frac{y}{m}} \frac{a(mh)a(mk)}{hk}$$
$$\cdot \Big(- \log \frac{Te^{2\gamma-1}}{2\pi hk} \sum_{p|k} p^{i\alpha} \log p - \sum_{p|k} p^{i\alpha} \log^2 p \Big) + O(TL^{(r+1)^2}). \tag{5.6}$$

This puts I_1 in a suitable form we need. We now simplify I_2 by substituting the Laurent expansions

$$(\zeta'/\zeta)(\overline{\tau}) = (i\alpha)^{-1} + O(1),$$
$$(\zeta'/\zeta)'(\overline{\tau}) = (i\alpha)^{-2} + O(1),$$
$$\zeta^2(\overline{\tau})\tau^{-1} = (i\alpha)^{-2} + (2\gamma - 1)(i\alpha)^{-1} + O(1)$$

in (5.5). The $O(1)$ terms of these laurent expansions contribute

$$TL\sum_{m\le y}\frac{1}{m}\sum_{n|m}\frac{1}{n}\sum_{h,k\le\frac{y}{m}}\frac{a_r(mh)a_r(mk)}{hk}\ll TL^{(r+1)^2}$$

by a calculation similar as in section 4, and

$$T\sum_{m\le y}\frac{1}{m}\sum_{n|m}1\sum_{h\le\frac{y}{m}}\frac{a(mh)}{h}\Big|\sum_{k\le\frac{y}{m}}a(mk)f(nk)\Big|\ll TL^{r^2+2r}$$

by Lemma 3.1, Lemma 3.10 and a calculation as before, for $f(k)=\mathcal{R}_k(1+i\alpha)/\phi(k)$ is multiplicative with $f(p^a)\ll p^{-a}$. Thus we deduce

$$\begin{aligned}I_2 =&\frac{T}{2\pi}\sum_{m\le y}\frac{1}{m}\sum_{n|m}\frac{\mu(n)}{n}\sum_{h,k\le\frac{y}{m}}\frac{a(mh)a(mk)}{hk}\\ &\cdot\Big(\frac{1+i\alpha\log\frac{T}{2\pi hkn^2}-(\frac{T}{2\pi hn})^{i\alpha}\frac{nk\mathcal{R}_{nk}(\tau)}{\phi(nk)}}{(i\alpha)^2}\Big)+O(TL^{(r+1)^2}).\end{aligned}\tag{5.7}$$

5.1. **Evaluation of I_1.** By (5.6) it follows that

$$\begin{aligned}I_1 =&\frac{T}{2\pi}\sum_{m\le y}\frac{\phi(m)}{m^2}\sum_{h,k\le\frac{y}{m}}\frac{a(mh)a(mk)}{hk}\\ &\times\Big(-\log\frac{Te^{2\gamma-1}}{2\pi hk}\sum_{p|k}p^{i\alpha}\log p-\sum_{p|k}p^{i\alpha}\log^2 p\Big)+O(TL^{(r+1)^2})\end{aligned}$$

with $a(n)=d_r(n)P_1(\frac{\log n}{\log y})+d_r^*(n)P_2(\frac{\log n}{\log y})$. As in section 4, we may replace $d_r^*(n)$ by $\frac{1}{\log^2 y}\sum_{p_1p_2|n}\mu^2(p_1p_2)\log p_1\log p_2$, and the overall error in evaluating I_1 caused by this is $O(L^{-1})$ smaller than the main term, which actually is $O(TL^{(r+1)^2})$. It follows that

$$I_1=\frac{T}{2\pi}\sum_{i_1=0,2}\sum_{i_2=0,2}\big(-L(a_{i_1,i_2,0,0,1})+a_{i_1,i_2,1,0,1}+a_{i_1,i_2,0,1,1}-a_{i_1,i_2,0,0,2}\big)\tag{5.8}$$

plus an error $O(TL^{(r+1)^2})$, where for $u,v,w\in\mathbb{Z}_{\ge0}$ we define $a_{i_1,i_2,u,v,w}$ to be the sum

$$\begin{aligned}&\frac{1}{(\log y)^{i_1+i_2}}\sum_{m\le y}\frac{\phi(m)}{m^2}\sum_{h\le\frac{y}{m}}\frac{P_{i_1}([mh]_y)\log^u h}{h}\\ &\cdot\sum_{p_1\cdots p_{i_1}|mh}\mu^2(p_1\cdots p_{i_1})\log p_1\cdots\log p_{i_1}d_r\Big(\frac{mh}{p_1\cdots p_{i_1}}\Big)\sum_{k\le\frac{y}{m}}\frac{P_{i_2}([mk]_y)\log^v k}{k}\\ &\cdot\sum_{q_1\cdots q_{i_2}|mk}\mu^2(q_1\cdots q_{i_2})\log q_1\cdots\log q_{i_2}d_r\Big(\frac{mk}{q_1\cdots q_{i_2}}\Big)\sum_{p|k}p^{i\alpha}\log^w p.\end{aligned}$$

Observe that $a_{i_1,i_2,u,v,w}$

$$\sim\frac{1}{(\log y)^{i_1+i_2}}\sum_{\substack{i_1'+i_1''=i_1\\ i_1',i_1''\geq 0}}\sum_{\substack{i_2'+i_2''=i_2\\ i_2',i_2''\geq 0}} C_{i_1}^{i_1'}C_{i_2}^{i_2'}\sum_{m\leq y}\frac{\phi(m)}{m^2}$$

$$\cdot\sum_{p_1\cdots p_{i_1'}|m}\mu^2(p_1\cdots p_{i_1'})\log p_1\cdots\log p_{i_1'}\sum_{q_1\cdots q_{i_2'}|m}\mu^2(q_1\cdots q_{i_2'})\log q_1\cdots\log q_{i_2'}$$

$$\cdot\sum_{h\leq\frac{y}{m}}\frac{P_{i_1}([mh]_y)\log^u h}{h}\sum_{p_{i_1'+1}\cdots p_{i_1'+i_1''}|h}\log p_{i_1'+1}\cdots\log p_{i_1'+i_1''}d_r\Big(\frac{mh}{p_1\cdots p_{i_1}}\Big)$$

$$\cdot\sum_{k\leq\frac{y}{m}}\frac{P_{i_2}([mk]_y)\log^v k}{k}\sum_{q_{i_2'+1}\cdots q_{i_2'+i_2''}|k}\log q_{i_2'+1}\cdots\log q_{i_2'+i_2''}d_r\Big(\frac{mk}{q_1\cdots q_{i_2}}\Big)$$

$$\cdot\sum_{p|k}p^{i\alpha}\log^w p \tag{5.9}$$

plus an error $O(L^{-1})$ smaller. For $p^{i\alpha}=\sum_{j=0}^{\infty}\frac{(i\alpha)^j}{j!}\log^j p$, the sum over k in (5.9) can be replaced by

$$\sum_{j=0}^{\infty}\frac{(i\alpha)^j}{j!}\Big(\sum_{k\leq\frac{y}{m}}\frac{P_{i_2}([mk]_y)(\log k)^v}{k}$$

$$\cdot\sum_{pq_{i_2'+1}\cdots q_{i_2'+i_2''}|k}(\log p)^{j+w}\log q_{i_2'+1}\cdots\log q_{i_2'+i_2''}d_r\Big(\frac{mk}{pq_1\cdots q_{i_2}}\Big)d_r(p)$$

$$+\,i_2''\sum_{k\leq\frac{y}{m}}\frac{P_{i_2}([mk]_y)(\log k)^v}{k}$$

$$\cdot\sum_{q_{i_2'+1}\cdots q_{i_2'+i_2''}|k}(\log q_{i_2'+1})^{j+w+1}\log q_{i_2'+2}\cdots\log q_{i_2'+i_2''}d_r\Big(\frac{mk}{q_1\cdots q_{i_2}}\Big)\Big).$$

Here, we ignore the terms with k contains square of p, for all these terms contribute $O(L^{-1})$ smaller than the main term in the calculation of $a_{i_1,i_2,u,v,w}$. Substituting this into (5.9), we denote

$$a_{i_1,i_2,u,v,w}=A_1+A_2 \tag{5.10}$$

plus an error $O(L^{-1})$ smaller with obvious meaning. Then a calculation similar to $\mathcal{M}_1$ in section 4 establishes

$$A_1\sim a_{r+1}\sum_{j=0}^{\infty}\frac{(i\alpha\log y)^j}{j!}(\log y)^{r^2+2r+u+v+w}$$

$$\cdot\sum_{\substack{i_1'+i_1''=i_1\\ i_1',i_1''\geq 0}}\sum_{\substack{i_2'+i_2''=i_2\\ i_2',i_2''\geq 0}} rc_r(i_1',i_2',i_1'',i_2'')\beta(w+j,r+i_2'')$$

$$\cdot\, l_{i_1,i_2}(i_1',i_2',i_1''+u,i_2''+v+w+j) \tag{5.11}$$

and

$$A_2 \sim a_{r+1} \sum_{j=0}^{\infty} \frac{(i\alpha \log y)^j}{j!} (\log y)^{r^2+2r+u+v+w}$$
$$\cdot \sum_{\substack{i_1'+i_1''=i_1 \\ i_1',i_1''\geq 0}} \sum_{\substack{i_2'+i_2''=i_2 \\ i_2',i_2''\geq 0}} i_2''(r+i_2''-1) c_r(i_1', i_2', i_1'', i_2'') \beta(w+j+1, r+i_2''-1)$$
$$\cdot\, l_{i_1,i_2}(i_1', i_2', i_1''+u, i_2''+v+w+j), \tag{5.12}$$

with $c_r(i_1', i_2', i_1'', i_2'')$ given by (1.13) and $l_{i_1,i_2}(\overrightarrow{n})$ given by (1.9). Whence, from (5.8), (5.21), (5.11) and (5.12), we obtain

$$I_1 \sim \frac{T}{2\pi} a_{r+1} L^{(r+1)^2+1} \sum_{j=0}^{\infty} \frac{z^j \eta^{j+(r+1)^2+1}}{j!}$$
$$\cdot \sum_{i_1=0,2} \sum_{i_2=0,2} \sum_{\substack{i_1'+i_1''=i_1 \\ i_1',i_1''\geq 0}} \sum_{\substack{i_2'+i_2''=i_2 \\ i_2',i_2''\geq 0}} c_r(i_1', i_2', i_1'', i_2'')(r h_{i_1,i_2}(i_1', i_2', i_1''+1, i_2''+1, j)$$
$$+ i_2''(r+i_2''-1) h_{i_1,i_2}(i_1', i_2', i_1''+1, i_2'', j+1)) \tag{5.13}$$

with h_{i_1,i_2} given by (1.11). Here, we have applied the formula $\beta(a,b) - \beta(a+1,b) = \beta(a,b+1)$ for $\forall a,b \geq 1$.

5.2. **Evaluation of I_2.** From (1.6), we recall that $a(n) = d_r(n)P_1(\frac{\log n}{\log y}) + d_r^*(n)P_2(\frac{\log n}{\log y})$ and use $\frac{1}{\log^2 y} \sum_{p_1 p_2 | n} \mu^2(p_1 p_2) \log p_1 \log p_2$ to replace $d_r^*(n)$ in (5.7) as before, then we may denote

$$I_2 = \frac{T}{2\pi} \sum_{i_1=0,2} \sum_{i_2=0,2} a'_{i_1,i_2} + O(TL^{(r+1)^2}) \tag{5.14}$$

with a'_{i_1,i_2} defined by the sum

$$\frac{1}{(\log y)^{i_1+i_2}} \sum_{m \leq y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n}$$
$$\sum_{h \leq \frac{y}{m}} \frac{P_{i_1}([mh]_y)}{h} \sum_{p_1 \cdots p_{i_1} | mh} \mu^2(p_1 \cdots p_{i_1}) \log p_1 \cdots \log p_{i_1} d_r\Big(\frac{mh}{p_1 \cdots p_{i_1}}\Big)$$
$$\sum_{k \leq \frac{y}{m}} \frac{P_{i_2}([mk]_y)}{k} \sum_{q_1 \cdots q_{i_2} | mk} \mu^2(q_1 \cdots q_{i_2}) \log q_1 \cdots \log q_{i_2} d_r\Big(\frac{mh}{q_1 \cdots q_{i_2}}\Big)$$
$$\Big(\frac{1 + i\alpha \log \frac{T}{2\pi hkn^2} - (\frac{T}{2\pi hn})^{i\alpha} \frac{nk\mathcal{R}_{nk}(\tau)}{\phi(nk)}}{(i\alpha)^2}\Big) \tag{5.15}$$

By an argument as before, we have

$$a'_{i_1,i_2} \sim \frac{1}{(\log y)^{i_1+i_2}} \sum_{\substack{i'_1+i''_1=i_1\\ i'_1,i''_1\geq 0}} \sum_{\substack{i'_2+i''_2=i_2\\ i'_2,i''_2\geq 0}} C_{i_1}^{i'_1} C_{i_2}^{i'_2} \sum_{m\leq y} \frac{1}{m} \sum_{n|m} \frac{\mu(n)}{n}$$

$$\sum_{p_1\cdots p_{i'_1}|m} \mu^2(p_1\cdots p_{i'_1}) \log p_1 \cdots \log p_{i'_1} \sum_{q_1\cdots q_{i'_2}|m} \mu^2(q_1\cdots q_{i'_2}) \log q_1 \cdots \log q_{i'_2}$$

$$\sum_{h\leq \frac{y}{m}} \frac{P_{i_1}([mh]_y)}{h} \sum_{p_{i'_1+1}\cdots p_{i'_1+i''_1}|h} \log p_{i'_1+1} \cdots \log p_{i'_1+i''_1} d_r\Big(\frac{mh}{p_1\cdots p_{i_1}}\Big)$$

$$\sum_{k\leq \frac{y}{m}} \frac{P_{i_2}([mk]_y)}{k} \sum_{q_{i'_2+1}\cdots q_{i'_1+i''_2}|k} \log q_{i'_1+1} \cdots \log q_{i'_1+i''_2} d_r\Big(\frac{mk}{q_1\cdots q_{i_2}}\Big)$$

$$\Big(\frac{1+i\alpha\log\frac{T}{2\pi hkn^2} - (\frac{T}{2\pi hn})^{i\alpha} \frac{nk\mathcal{R}_{nk}(\tau)}{\phi(nk)}}{(i\alpha)^2}\Big) \tag{5.16}$$

plus an error $O(L^{-1})$ smaller. Since all the terms with k that contains square of $q \in \{q_{i'_2+1}, \cdots, q_{i'_2+i''_2}\}$ contribute $O(L^{-1})$ smaller than the main term, we may ignore these terms in the following argument. Let $k = q_{i'_2+1}\cdots q_{i'_2+i''_2} k'$. For $f(k) = \mathcal{R}_k(1+i\alpha)/\phi(k)$ is multiplicative with $f(p^a) \ll p^{-a}$, we replace $\frac{\mathcal{R}_{nk}(\tau)}{\phi(nk)}$ by $f(q_{i'_2+1})\cdots f(q_{i'_2+i''_2})\frac{\mathcal{T}_{nk';r}(\alpha)}{\phi(nk')}$ with an error

$$\ll |\alpha|^{-2} L^r \sum_{m\leq y} \frac{d_r(m)}{m} \sum_{n|m} |\alpha|^{r+1} L^{2r} \frac{d_r(m) j_{\tau_0}(m)}{n^{1-\epsilon}} \ll L^{(r+1)^2}$$

in the calculation of a'_{i_1,i_2} by Lemma 3.1 and Lemma 3.11. A calculation shows that $\mathcal{R}_k(1) = \phi(k)/k$ and $\mathcal{R}'_k(1) = -\phi(k)\log k/k$, thus it follows that

$$\frac{\mathcal{T}_{nk';r}(\alpha)}{\phi(nk')} = \frac{1}{nk}(1 - i\alpha\log(nk')) + \sum_{j=2}^{r} \frac{\mathcal{R}^{(j)}_{nk'}(1)(i\alpha)^j}{\phi(nk')j!} \tag{5.17}$$

and

$$\begin{aligned} f(p) =& \frac{\mathcal{R}_p(\tau)}{\phi(p)} = \frac{1}{\phi(p)}(2 - p^{i\alpha} - \frac{1}{p^{1+i\alpha}}) \\ =& \frac{1}{p}(1 - i\alpha\log p) - \frac{1}{p}\sum_{j=2}^{\infty} \frac{(i\alpha)^j \log^j p}{j!} + O\Big(\frac{\log^2 p}{p^2}(i\alpha)^2\Big) \\ =& \frac{1}{p}\sum_{j=0}^{\infty} \frac{\Delta(j)(i\alpha)^j \log^j p}{j!} + O\Big(\frac{\log^2 p}{p^2}(i\alpha)^2\Big) \end{aligned} \tag{5.18}$$

with $\Delta(j)$ given by (1.15). Here, the $O(\frac{\log^2 p}{p^2}(i\alpha)^2)$ contributes $O(L^{(r+1)^2})$ in a'_{i_1,i_2} by a calculation as before. Substituting (5.17) and (5.18) into (5.16), we have the expressing

within the brackets of (5.16) simplifies to

$$-\sum_{u+j+j_1+\cdots+j_{i_2''}\geq 2}\frac{(\log\frac{T}{2\pi hn})^u\mathcal{R}_{nk'}^{(j)}(1)(i\alpha)^{u+j}\prod_{\tau=1}^{i_2''}\Big((i\alpha)^{j_\tau}\Delta(j_\tau)\log^{j_\tau}q_{i_2'+\tau}\Big)}{\phi(nk')u!\,j!\,j_1!\cdots j_{i_2''}!}$$

by replacing $(\frac{T}{2\pi hn})^{i\alpha}$ with $\sum_{u=0}^{\infty}\frac{(i\alpha)^u}{u!}(\log\frac{T}{2\pi hn})^u$. Employing this in (5.16), we have a'_{i_1,i_2} equal to

$$\begin{aligned}
&\frac{-1}{(\log y)^{i_1+i_2}}\sum_{\substack{i_1'+i_1''=2\\ i_1',i_1''\geq 0}}\sum_{\substack{i_2'+i_2''=2\\ i_2',i_2''\geq 0}}C_{i_1}^{i_1'}C_{i_2}^{i_2'}\\
&\cdot\sum_{u+j+j_1+\cdots+j_{i_2''}\geq 2}\frac{(i\alpha)^{u+j+j_1+\cdots+j_{i_2''}}\Delta(j_1)\cdots\Delta(j_{i_2''})}{u!\,j!\,j_1!\cdots j_{i_2''}!}\sum_{m\leq y}\frac{1}{m}\sum_{n|m}\frac{\mu(n)}{n}\\
&\cdot\sum_{p_1\cdots p_{i_1'}|m}\mu^2(p_1\cdots p_{i_1'})\log p_1\cdots\log p_{i_1'}\sum_{q_1\cdots q_{i_2'}|m}\mu^2(q_1\cdots q_{i_2'})\log q_1\cdots\log q_{i_2'}\\
&\cdot\sum_{h\leq\frac{y}{m}}\frac{P_{i_1}([mh]_y)}{h}\sum_{p_{i_1'+1}\cdots p_{i_1'+i_1''}|h}\log p_{i_1'+1}\cdots\log p_{i_1'+i_1''}d_r\Big(\frac{mh}{p_1\cdots p_{i_1}}\Big)\\
&\cdot\Big(\log\frac{T}{2\pi hn}\Big)^u\sum_{k\leq\frac{y}{m}}\frac{P_{i_2}([mk]_y)}{k}\sum_{q_{i_2'+1}\cdots q_{i_2'+i_2''}|k}\log^{j_1+1}q_{i_2'+1}\cdots\log^{j_{i_2''}+1}q_{i_2'+i_2''}\\
&\cdot d_r\Big(\frac{mk}{q_1\cdots q_{i_2}}\Big)\frac{nk'\mathcal{R}_{nk'}^{(j)}(1)}{\phi(nk')}
\end{aligned}\tag{5.19}$$

plus an error $O(L^{(r+1)^2})$. Whence, by Lemma 3.8 and Lemma 3.11, we have the sum over h in (5.19) equal to

$$\frac{\sigma_r(\frac{m}{p_1\cdots p_{i_1'}})(\log y)^{r+i_1''+u}}{(r+i_1''-1)!}\int_0^{1-[m]_y}\theta_1^{r+i_1''-1}(\eta^{-1}-\theta_1)^uP_{i_1}([m]_y+\theta_1)d\theta_1$$

and the sum over k equal to

$$\frac{\sigma_r(\frac{m}{q_1\cdots q_{i_2'}})(-1)^jC_r^j\,j_1!\cdots j_{i_2''}!(\log\frac{y}{m})^{r+i_2''+j+j_1+\cdots+j_{i_2''}}}{(r+i_2''+j+j_1+\cdots+j_{i_2''}-1)!}Q_{i_2,r+i_2''+j+j_1+\cdots+j_{i_2''}-1}([m]_y)$$

Employing these into (5.19), we interchange the order of the sum and the integration, and by Lemma 3.9, we have that

$$a'_{i_1,i_2} \sim (\log y)^{(r+1)^2} \sum_{\substack{i'_1+i''_1=i_1\\ i'_1,i''_1\geq 0}} \sum_{\substack{i'_2+i''_2=i_2\\ i'_2,i''_2\geq 0}} \sum_{u+j+j_1+\cdots+j_{i''_2}\geq 2} (i\alpha\log y)^{u+j+j_1+\cdots+j_{i''_2}-2}$$

$$\cdot \frac{C_{i_1}^{i'_1} C_{i_2}^{i'_2} (-1)^{j+1} C_r^j \Delta(j_1)\cdots\Delta(j_{i''_2}) a_{r+1} b_r(i'_1, i'_2)}{u!(r+i''_1-1)!(r+i''_2+j+j_1+\cdots+j_{i''_2}-1)!(r^2+i'_1+i'_2-1)!}$$

$$\cdot k_{i_1,i_2}(i''_1, u, i'_1+i'_2, i''_2+j+j_1+\cdots+j_{i''_2}) \tag{5.20}$$

plus an error $O(L^{(r+1)^2})$. We now simplify the expression of a'_{i_1,i_2} in three cases.

Case 1. $i''_2 = 0$. Replacing $j-2$ by n and $j+u-2$ by j in (5.20) respectively, we have

$$a'_{i_1,i_2} \sim (\log y)^{(r+1)^2} \sum_{\substack{i'_1+i''_1=i_1\\ i'_1,i''_1\geq 0}} \sum_{\substack{i'_2+i''_2=i_2\\ i'_2,i''_2\geq 0}} C_{i_1}^{i'_1} C_{i_2}^{i'_2} \sum_{j=0}^{\infty} (i\alpha\log y)^j$$

$$\cdot \sum_{n=-2}^{\min(r-2,j)} \frac{a_{r+1} b_r(i'_1, i'_2)(-1)^{n+1} C_r^{n+2}}{(j-n)!(r+i''_1-1)!(r+i''_2+n+1)!(r^2+i'_1+i'_2-1)!}$$

$$\cdot k_{i_1,i_2}(i''_1, j-n, i'_1+i'_2, i''_2+n+2) \tag{5.21}$$

Case 2. $i''_2 = 1$. We have

$$a'_{i_1,i_2} \sim (\log y)^{(r+1)^2} \sum_{\substack{i'_1+i''_1=i_1\\ i'_1,i''_1\geq 0}} \sum_{\substack{i'_2+i''_2=i_2\\ i'_2,i''_2\geq 0}} C_{i_1}^{i'_1} C_{i_2}^{i'_2} \sum_{j=0}^{\infty} (i\alpha\log y)^j$$

$$\cdot \sum_{n=-2}^{j} \frac{a_{r+1} b_r(i'_1, i'_2) \sum_{j'=-2}^{\min(r-2,n)} (-1)^{j'+1} C_r^{j'+2} \Delta(n-j')}{(j-n)!(r+i''_1-1)!(r+i''_2+n+1)!(r^2+i'_1+i'_2-1)!}$$

$$\cdot k_{i_1,i_2}(i''_1, j-n, i'_1+i'_2, i''_2+n+2) \tag{5.22}$$

by replacing $j-2$ with j', $j+j_1-2$ with n and $j+u+j_1-2$ with j in (5.20) respectively.

Case 3. $i''_2 = 2$. We have

$$a'_{i_1,i_2} \sim (\log y)^{(r+1)^2} \sum_{\substack{i'_1+i''_1=i_1\\ i'_1,i''_1\geq 0}} \sum_{\substack{i'_2+i''_2=i_2\\ i'_2,i''_2\geq 0}} C_{i_1}^{i'_1} C_{i_2}^{i'_2} \sum_{j=0}^{\infty} (i\alpha\log y)^j$$

$$\cdot \sum_{n=-2}^{j} \frac{a_{r+1} b_r(i'_1, i'_2) \sum_{j'=-2}^{\min(r-2,n)} (-1)^{j'+1} C_r^{j'+2} \sum_{j_1+j_2=n-j'} \Delta(j_1)\Delta(j_2)}{(j-n)!(r+i''_1-1)!(r+i''_2+n+1)!(r^2+i'_1+i'_2-1)!}$$

$$\cdot k_{i_1,i_2}(i''_1, j-n, i'_1+i'_2, i''_2+n+2) \tag{5.23}$$

by replacing $j-2$ with j', $j+j_1+j_2-2$ with n and $j+u+j_1+j_2-2$ with j in (5.20) respectively. Since

$$\sum_{j=0}^{r} (-1)^{j+1} C_r^j P(j) = 0$$

for any polynomial $P(j)$ on j, we have

$$\sum_{j'=-2}^{\min(r-2,n)} (-1)^{j'+1} C_r^{j'+2} \Delta(n-j') = 0 \qquad for \qquad n > r-2,$$

and

$$\sum_{j'=-2}^{\min(r-2,n)} (-1)^{j'+1} C_r^{j'+2} \sum_{j_1+j_2=n-j'} \Delta(j_1)\Delta(j_2) = 0 \quad for \quad n > r-2.$$

So, we simplify the expression of a'_{i_1,i_2} for all case to

$$a'_{i_1,i_2} \sim (\log y)^{(r+1)^2} a_{r+1} \sum_{j=0}^{\infty} (i\alpha \log y)^j \sum_{\substack{i'_1+i''_1=i_1 \\ i'_1,i''_1\geq 0}} \sum_{\substack{i'_2+i''_2=i_2 \\ i'_2,i''_2\geq 0}} c_r(i'_1, i'_2, i''_1, i''_2)$$

$$\cdot \sum_{n=-2}^{\min(r-2,j)} \frac{\Omega_r(i''_2, n)(r+i''_2-1)!}{(j-n)!(r+i''_2+n+1)!} k_{i_1,i_2}(i''_1, j-n, i'_1+i'_2, i''_2+n+2). \tag{5.24}$$

with $\Omega_r(i''_2, n)$ given by (1.14). Thus, substituting (5.24) into (5.14), we have

$$I_2 = \frac{T}{2\pi} L^{(r+1)^2+1} a_{r+1} \sum_{j=0}^{\infty} (z)^j \eta^{j+(r+1)^2+1} \sum_{i_1=0,2} \sum_{i_2=0,2} \sum_{\substack{i'_1+i''_1=i_1 \\ i'_1,i''_1\geq 0}} \sum_{\substack{i'_2+i''_2=i_2 \\ i'_2,i''_2\geq 0}} c_r(i'_1, i'_2, i''_1, i''_2)$$

$$\cdot \sum_{n=-2}^{\min(r-2,j)} \frac{\Omega_r(i''_2, n)(r+i''_2-1)!}{(j-n)!(r+i''_2+n+1)!} k_{i_1,i_2}(i''_1, j-n, i'_1+i'_2, i''_2+n+2) \tag{5.25}$$

plus an error $O(TL^{(r+1)^2})$. Theorem 1.3 follows from (1.22), (4.2),(5.13) and (5.25).

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Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, P. R. China.

E-mail address: fsj@amss.ac.cn

Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, P. R. China.

E-mail address: xswu@amss.ac.cn